

Factorial hypersurfaces in \mathbb{P}^4 with nodes

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Abstract. We prove that for $n = 5, 6, 7$ a nodal hypersurface of degree n in \mathbb{P}^4 is factorial if it has at most $(n-1)^2 - 1$ nodes.

Keywords : nodal hypersurface, factoriality, integral homology, base-point-freeness.

1. INTRODUCTION

Unless otherwise mentioned, every variety is always assumed to be projective, normal, and defined over \mathbb{C} . We also consider every divisor in the linear system $|\mathcal{O}_{\mathbb{P}^n}(k)|$ as a hypersurface in \mathbb{P}^n for simplicity.

A variety X is called factorial if each Weil divisor of X is Cartier. The factoriality is a very subtle property. It depends on both the local types of singularities and their global position. Also, it depends on the field of definition of the variety. In the present paper, we study the factoriality of a hypersurface in \mathbb{P}^4 . However, we confine our consideration to the case when they have only simple double points, *i.e.*, nodes.

Let V_n be a nodal hypersurface of degree n in \mathbb{P}^4 . Then the Picard group is isomorphic to the 2nd integral cohomology because $H^1(W, \mathcal{O}_W) = H^2(W, \mathcal{O}_W) = 0$ on a resolution W of V_n . In this case, the variety V_n is factorial if and only if the global topological property

$$\text{rank}(H^2(V_n, \mathbb{Z})) = \text{rank}(H_4(V_n, \mathbb{Z})),$$

holds. Note that the duality mentioned above fails on singular varieties in general. The nodes on V_n may have an effect on the integral (co)homology groups of V_n (See [6]). However, the rank of the 2nd integral cohomology group of V_n is 1 by Lefschetz theorem. Therefore, to determine whether the 3-fold hypersurface V_n is factorial or not, we have to see whether the rank of the 4th integral homology group of V_n is 1 or not. But, it is not simple to compute the rank of the 4th integral homology group of V_n . Fortunately, the paper [6] gives us a great method to compute the rank of the 4th integral homology group of V_n , which reduce the topological problem to a rather simple combinatorial problem. To be precise, the rank of the 4th integral homology group of V_n can be obtained by the following way:

Theorem 1.1. *Let V_n be a nodal hypersurface of degree n in \mathbb{P}^4 . The rank of the 4th integral homology group $H_4(V_n, \mathbb{Z})$ is equal to*

$$\#|\text{Sing}(V_n)| - I + 1,$$

where I is the number of independent conditions which vanishing on $\text{Sing}(V_n)$ imposes on homogeneous forms of degree $2n-5$ on \mathbb{P}^4 .

Proof. See [6]. □

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Therefore, the hypersurface V_n is factorial if and only if the set of nodes of the hypersurface V_n is $(2n - 5)$ -normal¹ in \mathbb{P}^4 (see [6]), in other words, the singular points of the hypersurface V_n impose linearly independent conditions on hypersurfaces of degree $2n - 5$ in \mathbb{P}^4 .

The geometry of the hypersurface V_n crucially depends on its factoriality. For example, in the case $n = 4$ the hypersurface V_n is non-rational whenever it is factorial (see [10]), which is not true without the factoriality condition.

Let us show an easy way to get a non-factorial hypersurface.

Example 1.2. Suppose that V_n is given by the equation

$$x_0g(x_0, x_1, x_2, x_3, x_4) + x_1f(x_0, x_1, x_2, x_3, x_4) = 0 \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]),$$

where g and f are general homogeneous polynomials of degree $n - 1$. Then the hypersurface V_n has exactly $(n - 1)^2$ nodes. They are located on the 2-plane defined by $x_0 = x_1 = 0$. The hypersurface V_n is not factorial because the hyperplane section $x_0 = 0$ splits into two non-Cartier divisors while the Picard group is generated by a hyperplane section (see [1]).

On the other hand, in the case when $\text{Sing}(V_n) < (n - 1)^2$, every smooth surface in V_n is cut by a hypersurface in \mathbb{P}^4 due to [5]. Therefore, it is natural that we should expect the following to be true.

Conjecture 1.3. *Every nodal hypersurface of degree n in \mathbb{P}^4 with at most $(n - 1)^2 - 1$ nodes is factorial.*

Conjecture 1.3 for $n = 2$ and 3 is trivial. For the case $n = 4$ Conjecture 1.3 is proved in [4]. In this paper we prove the following result.

Theorem 1.4. *Conjecture 1.3 holds for $n = 5, 6$, and 7 .*

Note that the following result is proved in [3].

Theorem 1.5. *A nodal hypersurface of degree n in \mathbb{P}^4 with at most $\frac{(n-1)^2}{4}$ nodes is factorial.*

Therefore, at least asymptotically Conjecture 1.3 is not far from being true. To our surprise, the conjecture below implies Conjecture 1.3.

Conjecture 1.6. *Let Σ be a subset in \mathbb{P}^4 such that at most $k(n - 1)$ points in Σ can be contained in a curve of degree k , where $|\Sigma| < (n - 1)^2$. Then at most $k(n - 1)$ points in $\phi_4(\Sigma)$ can be contained in a curve of degree k in \mathbb{P}^2 , where $\phi_4 : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ is a general projection.*

Unfortunately, we are unable to prove Conjecture 1.3 now, but we believe that the proof of Theorem 1.4 can help us to find new approaches to a proof of Conjecture 1.3.

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2. PRELIMINARIES

2.1. Projections and linear systems with zero-dimensional base loci. Let X be a smooth variety with a \mathbb{Q} -divisor $B_X = \sum_{i=1}^k a_i B_i$, where B_i is a prime divisor on X and a_i is a positive rational number. Let $\pi : Y \rightarrow X$ be a birational morphism of a smooth variety Y such that the union of all the proper transforms of the divisors B_i and all the π -exceptional divisors forms a divisor with simple normal crossing on Y .

¹In general, a subscheme X of \mathbb{P}^N is called d -normal in \mathbb{P}^N if the first cohomology of the sheaf of ideal of X twisted by $\mathcal{O}_{\mathbb{P}^N}(d)$ is zero. Throughout this paper, we consider a finite set of points in \mathbb{P}^n as a zero-dimensional reduced subscheme of \mathbb{P}^n .

Let B_Y be the proper transform of B_X on Y . Put $B^Y = B_Y - \sum_{i=1}^n d_i E_i$, where each E_i is an exceptional divisor of the morphism π and d_i is the rational number such that the equivalence

$$K_Y + B^Y \sim_{\mathbb{Q}} \pi^*(K_X + B_X)$$

holds. Then the log pair (Y, B^Y) is called the log pull back of the log pair (X, B_X) with respect to π , while the number d_i is called the discrepancy of the log pair (X, B_X) with respect to the π -exceptional divisor E_i .

By $\mathcal{L}(X, B_X)$ we denote the subscheme of the variety X associated to the ideal sheaf

$$\mathcal{I}(X, B_X) = \pi_* \left(\mathcal{O}_Y([-B^Y]) \right),$$

which is called the log canonical singularity subscheme of the log pair (X, B_X) .

We then obtain the following result due to [11].

Theorem 2.1. *Suppose that the divisor $K_X + B_X + H$ is numerically equivalent to a Cartier divisor for some nef and big \mathbb{Q} -divisor H on the variety X . Then for every $i > 0$ we have*

$$H^i \left(X, \mathcal{I}(X, B_X) \otimes \mathcal{O}_X(K_X + B_X + H) \right) = 0.$$

Theorem 2.1 gives us a useful tool to study the normality of a finite set in \mathbb{P}^N .

Lemma 2.2. *Let \mathcal{M} be a linear system consisting of hypersurfaces of degree k on \mathbb{P}^N . If the base locus Λ of the linear system \mathcal{M} is zero-dimensional, then the finite set Λ is $N(k-1)$ -normal in \mathbb{P}^N .*

Proof. Let H_1, \dots, H_r be general divisors in the linear system \mathcal{M} , where r is sufficiently big. We put

$$B = \frac{N}{r} \sum_{i=1}^r H_i.$$

Then the log pair (\mathbb{P}^N, B) is klt in the outside of the base locus Λ . For each point $p \in \Lambda$, we have $\text{mult}_p B \geq N$. Therefore, $\text{Supp}(\mathcal{L}(\mathbb{P}^N, B)) = \Lambda$.

Since the divisor $K_{\mathbb{P}^N} + B + H$ is \mathbb{Q} -linearly equivalent to $N(k-1)H$, where H is a hyperplane, we obtain $H^1(\mathbb{P}^N, \mathcal{I}(\mathbb{P}^N, B) \otimes \mathcal{O}_{\mathbb{P}^N}(N(k-1))) = 0$ from Theorem 2.1. Because $\text{Supp}(\mathcal{L}(\mathbb{P}^N, B)) = \Lambda$ and the scheme $\mathcal{L}(\mathbb{P}^N, B)$ is zero-dimensional, the set Λ that is the reduced scheme of $\mathcal{L}(\mathbb{P}^N, B)$ must be $N(k-1)$ -normal in \mathbb{P}^N . \square

Let Σ be a finite set of points in \mathbb{P}^N , $N \geq 3$, such that no $k(d-1) + 1$ points of Σ lie on a curve of degree k in \mathbb{P}^N for each $k \geq 1$, where $d \geq 3$ is a fixed integer. Fix a 2-plane Π in \mathbb{P}^N . We consider the projection

$$\phi_N : \mathbb{P}^N \dashrightarrow \Pi \cong \mathbb{P}^2$$

from a general $(N-3)$ -dimensional linear space L onto the 2-plane Π .

Lemma 2.3. *Let Λ be a subset of Σ and let \mathcal{M} be the linear system of hypersurfaces in \mathbb{P}^N of degree k that contains Λ . If $|\Lambda| > k(d-1)$ but the set $\phi_N(\Lambda)$ is contained in an irreducible curve on Π of degree k , then the base locus of the linear system \mathcal{M} is zero-dimensional.*

Proof. Suppose that the base locus of \mathcal{M} contains an irreducible curve Z . Let C be an irreducible curve on Π of degree k that contains $\phi_N(\Lambda)$ and let W be the cone in \mathbb{P}^N over the curve C with vertex L . Since W is a hypersurface of degree k in \mathbb{P}^N containing the set Λ , it belongs to the linear system \mathcal{M} . In particular, the curve Z is contained in the hypersurface W . Therefore, the curve Z is mapped onto the curve C because the linear space L is general and the curve C is irreducible. The curve Z has degree k because the restriction $\phi_N|_Z$ is a birational morphism to C .

If there is a point p in $\Lambda \setminus Z$, then the projection ϕ_N maps the point p to the outside of C because of the generality of the projection ϕ_N . Therefore, the set Λ must be contained in Z because

$\phi_N(\Lambda)$ is contained in C . However, the curve Z cannot contain more than $k(d-1)$ points of Σ . \square

Corollary 2.4. *A line on Π contains at most $d-1$ points of $\phi_N(\Sigma)$.*

Proof. It immediately follows from Lemma 2.3. \square

Corollary 2.5. *For $N = 3$, a curve of degree k on Π contains at most $k(d-1)$ points of $\phi_3(\Sigma)$ if $d \geq k^2 + 1$.*

Proof. For $k = 1$, it is true because of Corollary 2.4. Assume that the claim is true for $k < \ell$. We then suppose that there is a subset Λ of Σ such that $|\Lambda| > \ell(d-1)$ and the image $\phi_3(\Lambda)$ lie on a curve C of degree ℓ on Π . The curve C must be irreducible because of our assumption. Therefore, it follows from Lemma 2.3 that the base locus of the linear system \mathcal{M} of hypersurfaces of degree ℓ in \mathbb{P}^3 containing the set Λ is zero-dimensional. Let Q_1, Q_2 , and Q_3 be general members in \mathcal{M} . Then we obtain a contradictory inequality

$$\ell^3 = Q_1 \cdot Q_2 \cdot Q_3 > \ell(d-1) \geq \ell^3.$$

Therefore, for $d \geq k^2 + 1$, a curve of degree k on Π contains at most $k(d-1)$ points of $\phi_3(\Sigma)$. \square

Corollary 2.6. *For $N = 4$, a curve of degree k on Π contains at most $k(d-1)$ points of $\phi_4(\Sigma)$ if $d \geq k^2 + 1$.*

Proof. Let $\alpha : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from a generic point $o_1 \in \mathbb{P}^4$. We first claim that for $d \geq k^2 + 1$ no $k(d-1) + 1$ points of $\alpha(\Sigma)$ lie on a curve of degree k in \mathbb{P}^3 . It is obviously true for $k = 1$. Assume that the claim is true for $k < \ell$. And then we suppose that there is a subset Λ of $\ell(d-1) + 1$ points in Σ such that $\alpha(\Lambda)$ lie on a curve C of degree ℓ in \mathbb{P}^3 . The curve C must be irreducible. Let \mathcal{M} be the linear system of hypersurfaces of degree ℓ in \mathbb{P}^4 passing through all the points of Λ . Then the proof of Lemma 2.3 shows the base locus is zero-dimensional. Let W be the cone in \mathbb{P}^4 over the curve C with vertex o_1 . Then we get an absurd inequality

$$\ell^3 = Q_1 \cdot Q_2 \cdot W \geq |\Lambda| > \ell^3,$$

where each Q_i is a general member of \mathcal{M} . Therefore, at most $k(d-1)$ points of $\alpha(\Sigma)$ can lie on a curve of degree k in \mathbb{P}^3 if $d \geq k^2 + 1$.

For the projection $\beta : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from a generic point $o_2 \in \mathbb{P}^3$, we apply the proof of Corollary 2.5 to $\alpha(\Sigma)$. This completes our proof because the general projection ϕ_4 is the composite of the projections α and β . \square

2.2. Base-point-freeness. It is a classical result that if 6 points $p_1, \dots, p_6 \in \mathbb{P}^2$ in general position are blown up, then the complete linear system on the blow-up corresponding to $|\mathcal{O}_{\mathbb{P}^2}(3) - p_1 - \dots - p_6|$ is very ample as well as base-point-free. This is a key observation to classify del Pezzo surfaces.

Bese's paper [2] developed this observation to points on \mathbb{P}^2 in less general position and various divisors. The result however turned out to have a considerable generalization. E. Davis and A. Geramita obtained a very ampleness and a base-point-freeness theorems on blow-ups of \mathbb{P}^2 via the ideal-theoretic route that are more powerful than Bese's.

The theorem below is a special case of the paper [7] which provides a strong enough tool for us to study the base-point-freeness of linear systems of certain types on blow-ups of \mathbb{P}^2 .

Theorem 2.7. *Let $\pi : Y \rightarrow \mathbb{P}^2$ be the blow up at points p_1, \dots, p_s on \mathbb{P}^2 . Then the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^s E_i|$ is base-point-free for all $s \leq \max\{m(d+3-m)-1, m^2\}$, where $E_i = \pi^{-1}(p_i)$, $d \geq 3$, and $m = \lfloor \frac{d+3}{2} \rfloor$, if the set $\Gamma = \{p_1, \dots, p_s\}$ satisfies the following:*

no $k(d+3-k)-1$ points of Γ lie on a single curve of degree k , $1 \leq k \leq m$.

In the case $d = 3$ Theorem 2.7 is the well known result on the base-point-freeness of the anti-canonical linear system of a weak del Pezzo surface of degree $9 - s \geq 2$. The theorem above immediately implies the following:

Corollary 2.8. *Let $\Gamma = \{p_1, \dots, p_s\}$ be a finite set of points in \mathbb{P}^2 . For a given positive integer $d \geq 3$, if $s \leq \max\{m(d+3-m) - 1, m^2\}$ and no $k(d+3-k) - 1$ points of the set Γ lie on a curve of degree $k \leq m$ in \mathbb{P}^2 , where $m = \lfloor \frac{d+3}{2} \rfloor$, then for each point $p_i \in \Gamma$ there is a curve in \mathbb{P}^2 of degree d that contains all the points of the set Γ except the point p_i .*

The corollary above is the most important tool for this paper. Also, it makes us propose Conjecture 1.6.

2.3. Basic properties of nodes. As explained at the beginning, the ranks of the 4th homology groups of nodal hypersurfaces in \mathbb{P}^4 are strongly related to the number of nodes and their position. Even though the number of nodes are given in our problem, it is necessary that we should study their position.

Lemma 2.9. *Let V be a nodal hypersurface of degree n in \mathbb{P}^4 .*

- (1) *A curve of degree k in \mathbb{P}^4 contains at most $k(n-1)$ nodes of V .*
- (2) *If a 2-plane contains $\frac{n(n-1)}{2} + 1$ nodes of V , then the 2-plane is contained in V .*

Proof. Suppose that the hypersurface V is defined by an equation $F(x_0, x_1, x_2, x_3, x_4) = 0$. Then the singular locus of V is contained in a generic hypersurface $V' = (\sum \lambda_i \frac{\partial F}{\partial x_i} = 0)$ of degree $n-1$. Let C be a curve of degree k in \mathbb{P}^4 . Since the hypersurface V has only isolated singularities, the curve C cannot be contained in V' . Because the intersection number of the hypersurface V' and the curve C is $k(n-1)$, the curve C contains at most $k(n-1)$ singular points of V . Let Π be a 2-plane not contained in V . Then $V' \cap \Pi$ is a curve of degree $(n-1)$ not contained in V . Therefore, the curve $V' \cap \Pi$ cannot meet V at more than $\frac{n(n-1)}{2}$ nodes of V . \square

Lemma 2.10. *If a nodal hypersurface V of degree n in \mathbb{P}^4 contains a 2-plane, then it has at least $(n-1)^2$ singular points.*

Proof. Suppose that V contains a 2-plane Π . Then take two general hyperplanes H_1 and H_2 passing through the 2-plane Π . Then for each $i = 1, 2$ we have the residual surface Q_i of degree $n-1$ in H_i such that $H_i \cap V = \Pi \cup Q_i$. Then the $(n-1)^2$ intersection points of $Q_1 \cap Q_2 \cap \Pi$ are singular points of V . \square

2.4. Simple tools. To prove Theorem 1.4, we need various tools to handle hypersurfaces of certain degrees and a finite number of points in \mathbb{P}^4 .

Lemma 2.11. *Let $\Sigma = \{p_1, \dots, p_r\}$ be a set of r points in \mathbb{P}^N , $N \geq 2$. Let p be a point in $\mathbb{P}^N \setminus \Sigma$. Suppose that no $m+1$ points of Σ lie on a single line with the point p . Then there are at least $\min\{r-m, \lfloor \frac{r}{2} \rfloor\}$ mutually disjoint pairs of points in Σ such that each pair determines a line not containing the point p .*

Proof. We may assume that the points p_1, \dots, p_m , and p are on a single line L .

First, we suppose $m \geq r - m$. We then obtain $r - m$ such pairs by choosing one point from $\Sigma \cap L$ and the other from $\Sigma \setminus L$. Obviously, these pairs determine lines not passing through the point p .

Next, we suppose $m < r - m$. We can then find $\lfloor \frac{r-2m}{2} \rfloor$ such pairs by choosing two points of $\Sigma \setminus L$; otherwise $m+1$ points of Σ would lie on a single line. By choosing one point from the remaining points in $\Sigma \setminus L$ and the other from L we also obtain m such pairs. The number of the pairs that we obtain is $m + \lfloor \frac{r-2m}{2} \rfloor = \lfloor \frac{r}{2} \rfloor$. \square

Lemma 2.12. *Let H be a hyperplane in \mathbb{P}^N , $N \geq 3$ and X be a hypersurface of degree $d \geq 2$ in H . Suppose that the hypersurface X does not contain a point $o \in H$. For given two points p, q in $\mathbb{P}^N \setminus H$, there is a hypersurface of degree d in \mathbb{P}^N such that contains X and two points p and q but not the point o .*

Proof. Take a generic 2-plane Π passing through the points p and q in \mathbb{P}^N . Then Π meets X at at least two points, say p' and q' . Then the line determined by p and p' and the line determined by q and q' meet at a point v . Then the cone over X with vertex v is a hypersurface of degree d in \mathbb{P}^N containing X and two points p and q but does not contain the point o . \square

Lemma 2.13. *Let Λ and Δ be disjoint finite sets of points in \mathbb{P}^N , $N \geq 3$ and let p be a point in $\mathbb{P}^N \setminus (\Lambda \cup \Delta)$. Suppose that D_0 be a hypersurface of degree k in \mathbb{P}^N containing the set Λ but not the point p . If for each point $q \in \Delta$ there is a hypersurface D_q of degree k in \mathbb{P}^N containing the set $(\Lambda \cup \Delta \cup \{p\}) \setminus \{q\}$ but not the point q , then there is a hypersurface of degree k such that passes through $\Lambda \cup \Delta$ but not the point p in \mathbb{P}^N .*

Proof. Suppose that the hypersurface D_0 is defined by a homogeneous polynomial $g(x_0, \dots, x_N)$. Also, we suppose that each hypersurface D_q is defined by a homogeneous polynomial $f_q(x_0, \dots, x_N)$. Then $g(p) \neq 0$ and $f_q(p) = 0$ for each $q \in \Delta$. Furthermore, $f_q(q') \neq 0$ for some $q' \in \Delta$ if and only if $q = q'$. There is a complex number c_q for each $q \in \Delta$ such that $g(q) + c_q f_q(q) = 0$ because $f_q(q) \neq 0$. Then the hypersurface defined by

$$g(x_0, \dots, x_N) + \sum_{q \in \Delta} c_q f_q(x_0, \dots, x_N) = 0$$

contains the set $\Lambda \cup \Delta$ but not the point p . \square

Corollary 2.14. *Let \mathcal{M} be a linear system consisting of hypersurfaces of degree $k \geq 2$ on \mathbb{P}^N , $N \geq 3$. If the base locus Λ of the linear system \mathcal{M} is zero-dimensional, then for two distinct points p, q in $\mathbb{P}^N \setminus \Lambda$ and a point o in Λ , there is a hypersurface of degree $N(k-1)$ such that passes through $\Lambda \cup \{p, q\} \setminus \{o\}$ but not the point o in \mathbb{P}^N .*

Proof. By Lemma 2.2, there is a hypersurface D_0 of degree $N(k-1)$ in \mathbb{P}^N that passes through all the points of Λ except the point o . Let D be a general member in \mathcal{M} . We choose a hyperplane H_p in \mathbb{P}^N that passes through the point p but not the point q . We also choose a hyperplane H_q that passes through the point q but not the point p . We then apply Lemma 2.13 to the hypersurfaces D_0 , $D + (N(k-1) - k)H_p$, and $D + (N(k-1) - k)H_q$. \square

The result below is originally due to J. Edmonds ([8]). It can help us to make our proofs simpler.

Theorem 2.15. *Let Σ be a set of points in \mathbb{P}^N and let $d \geq 2$ be an integer. If no $dk + 2$ points of Σ lie in a k -plane for all $k \geq 1$, then the set Σ is d -normal in \mathbb{P}^N .*

Proof. See [9]. \square

3. CONJECTURAL PROOF

In this section, we prove Conjecture 1.3 under the assumption that Conjectures 1.6 is true. Let V_n be a nodal hypersurface of degree n in \mathbb{P}^4 . Suppose that $|\text{Sing}(V_n)| < (n-1)^2$ and $n \geq 4$. Fix a point $p \in \text{Sing}(V_n)$. And then put $\Gamma = \text{Sing}(V_n) \setminus \{p\}$. To prove the factoriality of V_n it is enough to construct a hypersurface of degree $2n-5$ that contains all the points of the set Γ and does not contain the point p .

We suppose that Conjecture 1.6 holds. Let $\phi_4 : \mathbb{P}^4 \dashrightarrow \Pi$ be the projection from a general line L in \mathbb{P}^4 , where Π is a 2-plane in \mathbb{P}^4 . It then follows from Conjecture 1.6 and Lemma 2.9 that the set $\phi_4(\Gamma)$ satisfies the condition

$$(3.1) \quad \text{no } k(n-1) + 1 \text{ points of } \phi_4(\Gamma) \text{ lie on a curve of degree } k \text{ on } \Pi \text{ for each } k \geq 1.$$

Remark 3.2. It follows from Corollaries 2.4 and 2.6 that the condition (3.1) holds for $k \leq \sqrt{n-1}$ without Conjecture 1.6.

Lemma 3.3. *For each $1 \leq k \leq n-1$, any curve of degree k on Π cannot contain $k(2n-2-k)-1$ points of $\phi_4(\Gamma)$.*

Proof. It is easy to check that $k(n-1) \leq k(2n-2-k)-2$ if $k < n$. \square

Lemma 3.4. *There is a curve of degree $2n-5$ on Π which passes through all the points of $\phi_4(\Gamma)$ but not the point $\phi_4(p)$.*

Proof. It immediately follows from Corollary 2.8 and Lemma 3.3. \square

Proposition 3.5. *Conjecture 1.6 implies Conjecture 1.3.*

Proof. Lemma 3.4 implies that there is a curve C of degree $2n-5$ on Π which passes through all the points of $\phi_4(\Gamma)$ but not the point $\phi_4(p)$. We take the cone over C with vertex L . The cone then contains all the points of Γ but not the point p . It implies that if the hypersurface V_n has $s < (n-1)^2$ singular points, then these s points impose s linearly independent conditions on homogeneous forms of degree $2n-5$ on \mathbb{P}^4 . Consequently, the rank of 4th singular homology group of V_n is 1 by Theorem 1.1, which completes the proof. \square

4. PROOF OF THEOREM 1.4

In this section, we prove Theorem 1.4. Let V_n be a nodal hypersurface of degree n in \mathbb{P}^4 with at most $(n-1)^2-1$ nodes. However, as we will see in the proofs, we may assume that the hypersurface V_n has exactly $(n-1)^2-1$ nodes. To prove the factoriality of the hypersurface V_n , for an arbitrary point $p \in \text{Sing}(V_n)$, we have to construct a hypersurface of degree $(2n-5)$ in \mathbb{P}^4 that contains the set $\text{Sing}(V_n)$ except the point p .

4.1. Quintic hypersurfaces. Let V_5 be a nodal quintic hypersurface in \mathbb{P}^4 with 15 nodes. A line can contain at most 4 nodes by Lemma 2.9. If a 2-plane Π contains 12 nodes of V_5 , then Π is contained in V_5 by Lemma 2.9. It then follows from Lemma 2.10 that V_5 must have at least 16 nodes, which contradicts our assumption. Therefore, a 2-plane can contain at most 11 nodes of V_5 . Therefore, the set of nodes of V_5 satisfies the condition for $d=5$ in Theorem 2.15 and hence the nodal quintic hypersurface V_5 is factorial.

4.2. Sextic hypersurfaces. Let V_6 be a nodal sextic hypersurface in \mathbb{P}^4 with 24 nodes. We denote the set of nodes of V_6 by Σ . If a 2-plane Π contains 16 nodes of V_6 , then Π is contained in V_6 by Lemma 2.9. It then follows from Lemma 2.10 that V_6 must have at least 25 nodes, which contradicts our assumption. Therefore, a 2-plane contains at most 15 nodes of V_6 .

Proposition 4.1. *If no 23 points of Σ lie on a single 3-plane, then the hypersurface V_6 is factorial.*

Proof. Since no 6 points of Σ lie on a single line and no 16 points of Σ lie on a single 2-plane, the set Σ satisfies the condition for $d=7$ of Theorem 2.15. Therefore, the set Σ is 7-normal in \mathbb{P}^4 and hence the hypersurface V_6 is factorial. \square

Pick an arbitrary point p in Σ and then we denote the set $\Sigma \setminus \{p\}$ by Γ . To prove the factoriality of V_6 , we must find a hypersurface of degree 7 in \mathbb{P}^4 that contains the set Γ but not the point p . Due to Proposition 4.1, we may assume that at least 23 points of Σ lie in a single 3-plane H . Furthermore, we may assume that all the 24 points of Σ lie in the 3-plane H because in what follows we will show that there is a septic hypersurface in H , not in \mathbb{P}^4 , that contains $\Gamma \cap H$ but not the point p .

We consider the projection $\phi_3 : H \dashrightarrow \Pi$ from a generic point o in H , where Π is a generic hyperplane of H . At most 5 points of Σ can lie on a single line in H and at most 10 points of Σ can lie on a conic on H .

Lemma 4.2. *If there is a set Λ of at least 20 points of Γ such that $\phi_3(\Lambda)$ is contained in a cubic curve C on Π , then there is a septic hypersurface in H that contains the set Γ but not the point p .*

Proof. We may assume that the cubic curve C contains the point $\phi_3(p)$. If not, then we can easily construct a septic surface in H that contains Γ but not the point p . The curve C must be irreducible because a line (a conic, resp.) contains at most 5 (10, resp.) points of $\phi_3(\Lambda)$ by Corollaries 2.4 and 2.5. It then follows from Lemma 2.3 that the linear system of cubic surfaces in H passing through $\Lambda \cup \{p\}$ has zero-dimensional base locus. Therefore, applying Corollary 2.14, we obtain a sextic surface F that passes through 22 points of Σ but not the point p . Note that $|\Gamma \setminus F| \leq 1$. By taking a general hyperplane passing through the point in $\Gamma \setminus F$, we can construct a septic surface in H that contains the set Γ but not the point p . \square

From now, we suppose that no 20 points of $\phi_3(\Lambda)$ is contained in a cubic curve on Π . And then let us apply the similar technique as Lemma 2.2, which has evolved from the papers [4] and [12], to the following case.

Lemma 4.3. *If the set $\phi_3(\Lambda)$ is contained in a quartic curve C on Π , then there is a septic hypersurface in H that contains the set Γ but not the point p .*

Proof. The curve C must be irreducible because of our assumption. Also, we may assume that it contains the point $\phi_3(p)$ as well. Then the linear system \mathcal{M} of quartic hypersurfaces in H passing through Σ has zero-dimensional base locus. Meanwhile, we have the sextic surface $Y = H \cap V_6$ contains all the nodes of V_6 . It may have non-isolated singularities. However, it is irreducible and reduced; otherwise the hypersurface V_6 would have more than 24 nodes. Choose a general enough surface S' in \mathcal{M} . Then it is smooth in the outside of the base locus of \mathcal{M} and hence it is normal. Also, the surface Y gives us a reduced divisor $D_6 \in |\mathcal{O}_{S'}(6)|$ on S' . Let D_4 be a divisor in $|\mathcal{O}_{S'}(4)|$ given by a general member of \mathcal{M} . We then consider the \mathbb{Q} -divisor $D = (1 - \epsilon)D_6 + 2\epsilon D_4$, where ϵ is sufficiently small enough rational number. Then it is easy to check that the support of $\mathcal{L}(S', D)$ is zero-dimensional and contains Σ . Use Theorem 2.1 to obtain $H^1(S', \mathcal{I}(S', D) \otimes \mathcal{O}_{S'}(7)) = 0$. Therefore, there is a divisor in $|\mathcal{O}_{S'}(7)|$ that contains Γ but not the point p . The exact sequence

$$0 \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(7)) \rightarrow H^0(S', \mathcal{O}_{S'}(7)) \rightarrow 0$$

completes the proof. \square

Lemma 4.4. *If the set $\phi_3(\Gamma)$ is not contained in any quartic curve C on Π , then there is a septic hypersurface in H that contains the set Γ but not the point p .*

Proof. In this case, the set $\phi_3(\Lambda)$ satisfies the condition for $d = 7$ in Theorem 2.8. Therefore, there is a septic curve on Π that contains the set $\phi_3(\Lambda)$ but not the point $\phi_3(p)$. Then the cone over the septic curve with vertex o is a septic hypersurface in H that contains Γ but not the point p . \square

Consequently, for an arbitrary point $p \in \Sigma$, we can find a septic hypersurface in \mathbb{P}^4 that contains Γ but not the point p . Therefore, the rank of the 4th integral homology group of V_6 is 1 and hence V_6 is factorial.

4.3. Septic hypersurfaces. Let V_7 be a nodal septic hypersurface in \mathbb{P}^4 with 35 nodes. We again denote the set of nodes of V_7 by Σ . Fix a point p in Σ . We denote the set $\Sigma \setminus \{p\}$ by Γ . To prove the factoriality of V_7 , we have to construct a hypersurface of degree 9 in \mathbb{P}^4 that contains the set Γ but not the point p .

First of all, it follows from Lemma 2.10 that a 2-plane contains at most 21 points of Σ . Suppose that there is a 2-plane Π containing at least 20 points of Σ . The 2-plane Π is not contained in V_7 ; otherwise the hypersurface V_7 would have at least 36 nodes. We then consider the projection

$\phi_4 : \mathbb{P}^4 \dashrightarrow \Pi$ from a generic line L . By Corollaries 2.4 and 2.6 a line on Π contains at most 6 points of $\phi_4(\Sigma)$ and a conic on Π contains at most 12 points of $\phi_4(\Sigma)$.

Proposition 4.5. *If there is a 2-plane Π containing at least 20 points of Σ , then for the projection $\phi_4 : \mathbb{P}^4 \dashrightarrow \Pi$ from a generic line L the set $\phi_4(\Gamma)$ satisfies the following:*

- (1) *A line on Π contains at most 6 points of $\phi_4(\Gamma)$.*
- (2) *A conic on Π contains at most 12 points of $\phi_4(\Gamma)$.*
- (3) *A cubic on Π contains at most 25 points of $\phi_4(\Gamma)$.*
- (4) *A quartic on Π contains at most 30 points of $\phi_4(\Gamma)$.*
- (5) *A quintic on Π contains at most 33 points of $\phi_4(\Gamma)$.*
- (6) *A sextic on Π contains at most 34 points of $\phi_4(\Gamma)$.*

Proof. The first and the second statements follow from Corollaries 2.4 and 2.6. And the last statement is obvious because $|\phi_4(\Gamma)| = 34$.

For a cubic, we suppose that there is a cubic C on Π that contains 26 points $\phi_4(p_1), \dots, \phi_4(p_{26})$ of $\phi_4(\Gamma)$. The cubic C must be irreducible because of the first and the second statements. It then follows from Lemma 2.3 that the base locus of the linear system \mathcal{M} of cubic hypersurfaces in \mathbb{P}^4 containing the points p_1, \dots, p_{26} is zero-dimensional and hence the restricted linear system $\mathcal{M}|_\Pi$ of the linear system \mathcal{M} to the 2-plane Π also has zero-dimensional base locus. Since we have at most 15 points of Σ in the outside of Π , at least 11 points of p_1, \dots, p_{26} belong to Π . Therefore, there is an irreducible cubic curve D in $\mathcal{M}|_\Pi$ that is not contained in V_7 but passing through 11 nodes of V_7 . However, this is impossible because $21 = D \cdot V_7 \geq 11 \cdot 2 = 22$.

If we have a quartic in Π containing 31 points of $\phi_4(\Gamma)$, then in the same way, we can find an irreducible quartic curve not contained in V_7 but passing through 16 nodes of V_7 , which is absurd.

Finally, if there is a quintic in Π containing 34 points of $\phi_4(\Gamma)$, then in the same way we can find an irreducible quintic curve not contained in V_7 but passing through 19 nodes of V_7 , which is also impossible. \square

Corollary 4.6. *If there is a 2-plane containing at least 20 points of Σ , the hypersurface V_7 is factorial.*

Proof. The proposition above shows the set $\phi_4(\Gamma)$ satisfies the condition for $d = 9$ in Corollary 2.8 and hence there is a curve C of degree 9 on Π passing through all the point of $\phi_4(\Gamma)$ but not the point $\phi_4(p)$. The cone over the curve C with vertex L shows that the set Σ is 9-normal in \mathbb{P}^4 . Therefore, the hypersurface V_7 is factorial. \square

From now on, we suppose that a 2-plane contains at most 19 points of Σ . If a hyperplane in \mathbb{P}^4 contains at most 28 points of Σ , then the set Σ satisfies the condition for $d = 9$ in Theorem 2.15 and hence it is 9-normal and V_7 is factorial. Therefore, we suppose that a hyperplane H in \mathbb{P}^4 contains at least 29 points of Σ . And let $\Sigma' = \Sigma \cap H$ and $\Sigma'' = \Sigma \setminus H$. We always assume that the point p is contained in Σ' because, if not, then we can easily construct a hypersurface of degree 9 in \mathbb{P}^4 containing Σ except the point p .

We consider the projection $\alpha : H \dashrightarrow \Pi$ from a generic point $o_1 \in H$, where Π is a general 2-plane in H .

Lemma 4.7. *If there is a set Λ of at least 26 points of $\Sigma' \setminus \{p\}$ such that $\alpha(\Lambda)$ is contained in a cubic curve C on Π , then there is a hypersurface of degree 9 that contains the set Γ but not the point p .*

Proof. Note that the curve C is irreducible and $m = |\Gamma \setminus \Lambda| \leq 8$.

Suppose that the curve C does not contain the point $\alpha(p)$. Because a line has at most 6 points of Σ , there is a hypersurface of degree $\min\{m - 5, \lfloor \frac{m}{2} \rfloor\} \leq 4$ in \mathbb{P}^4 that passes through $\Gamma \setminus \Lambda$ but not the point p by Lemma 2.11. Therefore, we can easily construct a hypersurface of degree 9 that passes through Γ but not the point p .

Suppose that the curve C contains also the point $\alpha(p)$. Pick two points p_1 and p_2 from $\Gamma \setminus \Lambda$ in such a way that $\Gamma \setminus (\Lambda \cup \{p_1, p_2\})$ is contained in a cubic hypersurface F_1 in \mathbb{P}^4 not containing the point p , which is possible because of Lemma 2.11. The linear system of cubic hypersurfaces in H containing $\Lambda \cup \{p\}$ has zero dimensional base locus. Therefore, there is a sextic hypersurface F_2 in \mathbb{P}^4 that passes through Λ and the points p_1 and p_2 but not the point p by Corollary 2.14. Then the nonic hypersurface $F = F_1 + F_2$ contains all the point of Σ except the point p . \square

We may assume that no 26 points of $\alpha(\Sigma' \setminus \{p\})$ lie on a cubic curve on Π . If a cubic curve on Π contains more than 18 points of $\alpha(\Sigma' \setminus \{p\})$ then it must be irreducible.

Lemma 4.8. *If a cubic curve C on Π contains 22 points of $\alpha(\Sigma' \setminus \{p\})$, it is unique.*

Proof. Suppose that a cubic curve C' on Π contains at least 22 points of $\alpha(\Sigma' \setminus \{p\})$, then it meets C at $22 - (34 - 22) = 10$ points and hence $C = C'$. \square

To prove the factoriality of V_7 , we will consider the following five cases.

Case 1. $|\Sigma'| = 29$.

Because no 22 points of Σ are contained in a 2-plane, Lemma 4.8 enables us to choose six points p_1, p_2, \dots, p_6 from the set $\Sigma' \setminus \{p\}$ in such a way that

- no 20 points of $\alpha(\Sigma' \setminus \{p, p_1, \dots, p_6\})$ lie on a single cubic on Π ;
- p_1, p_2 , and p_3 lie on a 2-plane Π_1 not containing the point p ;
- p_4, p_5 , and p_6 lie on a 2-plane Π_2 not containing the point p .

Then the set $\alpha(\Sigma' \setminus \{p, p_1, \dots, p_6\})$ satisfies the condition of Corollary 2.8 for $d = 7$. Therefore, there is a septic curve D on Π containing $\alpha(\Sigma' \setminus \{p, p_1, \dots, p_6\})$ but not the point p . Then the cone over D with vertex o_1 is the septic surface \bar{D} in H containing $\Sigma' \setminus \{p, p_1, \dots, p_6\}$ but not the point p . Choose two points q_1 and q_2 from Σ'' . By Lemma 2.12, we can construct a hypersurface \tilde{D} in \mathbb{P}^4 containing $\Sigma' \setminus \{p, p_1, \dots, p_6\}$ and q_1 and q_2 but not the point p . Now we choose another two points q_3 and q_4 from Σ'' . Let H_3 be a hyperplane passing through the point q_3 but not q_4 and H_4 a hyperplane passing through the point q_4 but not q_3 . Apply Lemma 2.13 to the hypersurfaces \tilde{D} , $H + H' + 5H_4$, and $H + H' + 5H_3$, where H' is a general hyperplane in \mathbb{P}^4 passing through q_1 and q_2 , to obtain a septic hypersurface in \mathbb{P}^4 passing through $\Sigma' \setminus \{p, p_1, \dots, p_6\}$ and $\{q_1, q_2, q_3, q_4\}$ but not the point p . By our construction, two 2-planes Π_1 and Π_2 and the remaining two points of Σ'' are contained in a quadratic hypersurface in \mathbb{P}^4 not passing through the point p .

Case 2. $|\Sigma'| = 30$.

For the same reason as in Case 1, we can choose three points p_1, p_2 , and p_3 from the set $\Sigma' \setminus \{p\}$ in such a way that

- no 23 points of $\alpha(\Sigma' \setminus \{p, p_1, p_2, p_3\})$ lie on a single cubic on Π ;
- p_1, p_2 , and p_3 lie on a 2-plane Π_1 not containing the point p .

Then the set $\alpha(\Sigma' \setminus \{p, p_1, p_2, p_3\})$ satisfies the condition of Corollary 2.8 for $d = 8$. Therefore, there is an octic curve D on Π containing $\alpha(\Sigma' \setminus \{p, p_1, p_2, p_3\})$ but not the point p . Then the cone over D with vertex o_1 is an octic surface \bar{D} in H containing $\Sigma' \setminus \{p, p_1, p_2, p_3\}$ but not the point p . Choose two points q_1 and q_2 from Σ'' . By Lemma 2.12, we can construct a hypersurface \tilde{D} in \mathbb{P}^4 containing $\Sigma' \setminus \{p, p_1, p_2, p_3\}$ and q_1 and q_2 . Now we choose another two points q_3 and q_4 from Σ'' . As in Case 1, we use Lemma 2.13 to get an octic hypersurface in \mathbb{P}^4 passing through $\Sigma' \setminus \{p, p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3, q_4\}$ but not the point p . Also, the 2-plane Π_1 and the remaining one point Σ'' are contained in a hyperplane in \mathbb{P}^4 not passing through the point p .

Case 3. $|\Sigma'| = 31$.

Then the set $\alpha(\Sigma' \setminus \{p\})$ satisfies the condition of Corollary 2.8 for $d = 9$. Therefore, there is a nonic curve D on Π containing $\alpha(\Sigma' \setminus \{p\})$ but not the point p . Then the cone over D with vertex o_1 is a nonic surface \bar{D} in H containing $\Sigma' \setminus \{p\}$ but not the point p . Choose two points q_1 and q_2 from Σ'' . By Lemma 2.12, we can construct hypersurface \tilde{D} in \mathbb{P}^4 containing $\Sigma' \setminus \{p, p_1, p_2, p_3\}$ and q_1 and q_2 . Note that $|\Sigma'' \setminus \{q_1, q_2\}| = 2$. As in the previous, we use Lemma 2.13 to construct a nonic hypersurface in \mathbb{P}^4 passing through Γ but not the point p .

Case 4. $32 \leq |\Sigma'| \leq 34$.

Suppose that no 31 points of $\alpha(\Sigma' \setminus \{p\})$ lie on a single quartic curve on Π . The set $\alpha(\Sigma' \setminus \{p\})$ then satisfies the condition of Corollary 2.8 for $d = 9$. Therefore, as in Case 3, we can find a nonic hypersurface that we need.

Suppose that there is a set Λ of at least 31 points of $\Sigma' \setminus \{p\}$ such that a quartic curve C on Π contains $\alpha(\Lambda)$. If the curve C does not contain the point $\alpha(p)$, then we can easily construct a nonic hypersurface in \mathbb{P}^4 that we are looking for. Therefore, we may assume that the point $\alpha(p)$ also belongs to C . Then it follows from Lemmas 2.2 and 2.3 that there is a nonic hypersurface in H passing through Λ but not the point p . Then, using Lemmas 2.12 and 2.13, we can construct a nonic hypersurface in \mathbb{P}^4 that passes through all the point of Γ but not the point p .

Case 5. $|\Sigma'| = 35$.

Suppose that there is a set Λ of at least 31 points of $\Gamma = \Sigma' \setminus \{p\}$ such that $\alpha(\Lambda)$ is contained in a quartic curve C on Π . The curve C must be irreducible. We also assume that the curve C contains the point $\alpha(p)$. Then the base locus of the linear system \mathcal{M} of quartic surfaces on H containing $\Lambda \cup \{p\}$ is zero-dimensional by Lemma 2.3. Let B be the support of the base locus of the linear system \mathcal{M} and $\bar{\Sigma} = \Sigma \setminus B$. Note that $\Lambda \cup \{p\} \subset B$. It follows from Lemma 2.2 that the set B is 9-normal in H . There is a nonic hypersurface F in H that contains $B \setminus \{p\}$ but not the point p . Because $|\bar{\Sigma}| \leq 3$, for each $q \in \bar{\Sigma}$ there is a quintic hypersurface Q_q in H such that contains the set $\bar{\Sigma}$ but not the point q . Choose a general element Q from the linear system \mathcal{M} . We then apply Lemma 2.13 to the nonic hypersurfaces F and $Q + Q_q$ to obtain a nonic hypersurface passing through Σ except the point p . Therefore, we may assume that no 31 points of $\alpha(\Gamma)$ lie on a quartic on Π .

Unless the set $\alpha(\Gamma)$ lie on a quintic curve on Π , we can use Corollary 2.8 to get a nonic curve on Π containing the set $\alpha(\Gamma)$ but not the point $\alpha(p)$, which gives us a nonic hypersurface in \mathbb{P}^4 that we need.

Finally, we suppose that there is a quintic curve C_5 on Π that contains $\alpha(\Gamma)$.

The curve C_5 is irreducible. Also, we may assume that it contains the point $\alpha(p)$ as well. Then the linear system \mathcal{D} of quintic hypersurfaces in H passing through Σ has zero-dimensional base locus. Meanwhile, we have the septic surface $Y = H \cap V_7$ contains all the nodes of V_7 , which may have non-isolated singularities. However, it is irreducible and reduced; otherwise the hypersurface would have more than 35 nodes. Choose a general enough surface S' in \mathcal{D} . Then it is smooth in the outside of the base locus of \mathcal{D} and hence it is normal. Also, the surface Y gives us a reduced divisor $D_7 \in |\mathcal{O}_{S'}(7)|$ on S' . Let D_5 be a divisor in $|\mathcal{O}_{S'}(5)|$ given by a general member of \mathcal{D} . We then consider the \mathbb{Q} -divisor $D = (1 - \epsilon)D_7 + 2\epsilon D_5$, where ϵ is sufficiently small enough rational number. Then it is easy to check that the support of $\mathcal{L}(S', D)$ is zero-dimensional and contains Σ . Using Theorem 2.1, we obtain $H^1(S', \mathcal{I}(S', D) \otimes \mathcal{O}_{S'}(9)) = 0$. Therefore, there is a divisor in $|\mathcal{O}_{S'}(9)|$ that contains Γ but not the point p . Because the sequence

$$0 \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(9)) \rightarrow H^0(S', \mathcal{O}_{S'}(9)) \rightarrow 0$$

exact, we are done.

REFERENCES

- [1] A. Andreotti, Th. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. Math. **69** (1959), 713–717.
- [2] E. Bese, *On the spannedness and very ampleness of certain line bundles on the blow-ups of $\mathbb{P}_{\mathbb{C}}^2$ and \mathbb{F}_r* , Math. Ann. **262** (1983), 225–238.
- [3] I. Cheltsov, *On factoriality of nodal threefolds*, Jour. Alg. Geometry **14** (2005), 663–690.
- [4] I. Cheltsov, *Non-rational nodal quartic threefolds*, Pacific J. of Math., to appear.
- [5] C. Ciliberto, V. di Gennaro, *Factoriality of certain hypersurfaces of \mathbb{P}^3 with ordinary double points*, Encyclopaedia of Mathematical Sciences **132** Springer-Verlag, Berlin, (2004), 1–9.
- [6] S. Cynk, *Defect of a nodal hypersurface*, Manuscripta Math. **104** (2001), 325–331.
- [7] E. Davis, A. Geramita, *Birational morphisms to \mathbb{P}^2 : an ideal-theoretic perspective*, Math. Ann. **279** (1988), 435–448.
- [8] J. Edmonds, *Minimum partition of a matroid into independent subsets*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 67–72.
- [9] D. Eisenbud, J-H. Koh, *Remarks on Points in a Projective Space*, Commutative Algebra, Berkeley, CA, (1987), Mathematical Science Research Institute Publications **15**, Springer, New York 157–172.
- [10] M. Mella, *Birational geometry of quartic 3-folds II: the importance of being \mathbb{Q} -factorial*, Math. Ann. **330** (2004), 107–126.
- [11] V. Shokurov, *Three-dimensional log perestroikas*, Izv. Ross. Akad. Nauk **56** (1992), 105–203.
- [12] J. Park, Y. Woo, *A remark on hypersurfaces with isolated singularities*, preprint (2005).